COMPUTATION OF A LAVAL NOZZLE

(O RASCHETE SOPLA LAVALIA)

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A. Sh. DORFMAN
(Kiev)

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Frankl' showed in his article [1] that the main term in Chaplygin's equations can be used for the computation of "shock-free" plane-parallel Laval nozzles in the region close to the throat. Frankl' presented the main term of the solution in a form of a linear combination of two hypergeometric functions. Later Fal'kovich succeeded in transforming the main term into a polynomial of third degree, thus simplifying the analysis and computation of the flow in the nozzle. Both solutions are sufficiently accurate, however, only in a zone immediately close to the throat of the nozzle. This restricts their practical applicability.

This article presents an approximate solution of the equations of Chaplygin, which are more accurate at a much larger distance from the nozzle throat, than the solutions by Frankl' and Fal'kovich. The solution obtained is applied to the construction of the initial portion of the supersonic part of a Laval nozzle, whose streamlines in the subsonic part have the form [1]:

$$\psi = -B_1\theta - B_2 \sum_{n=1}^{\infty} \frac{Z_{Bn}(\tau)}{Z_{Bn}(\tau^*)} \frac{\sin 2Bn\,\theta}{n^{4/3}} \qquad \left(\tau = \frac{v^2}{\frac{k+1}{k-1}\, a_*^2}\right) \tag{0.1}$$

Here B, B_1, B_2 are constants, Z_{B_n} is Chaplygin's function, τ Chaplygin's variable, v the flow velocity, a_* the critical velocity, k the adiabatic exponent, τ^* the value of τ for $v = a_*$.

1. Integration of an auxiliary system of equations. To solve the problem we shall proceed from Chaplygin's equations [2]:

$$\frac{\partial\theta}{\partial\varphi} = -\frac{1}{b}\frac{\partial\eta}{\partial\psi}, \qquad \frac{\partial\theta}{\partial\psi} = b\eta\frac{\partial\eta}{\partial\varphi} \tag{1.1}$$

where

$$b = \frac{\rho_0}{\rho}\sqrt{\frac{1-M^2}{\eta}}, \qquad \eta = \left(\frac{3}{2}\int_{\bar v}^{1} \frac{\sqrt{1-M^2}}{\bar v}\, d\bar v\right)^{2/3}, \qquad M = \frac{v}{a}, \qquad \bar v = \frac{v}{a_*} \tag{1.2}$$

Here θ is the angle of inclination of the velocity vector, ρ the gas density, ρ_0 the stagnation density , a the sound velocity in the gas flow, ψ the stream function, ϕ the velocity potential.

An auxiliary system of equations will be set up for the solution of the proposed problem. The function b in the system (1.1), given by equation (1.2), is replaced by an arbitrary function h. The system (1.1) will then become

$$\frac{\partial\theta}{\partial\varphi} = -\frac{1}{h}\frac{\partial\eta}{\partial\psi}, \qquad \frac{\partial\theta}{\partial\psi} = h\eta\frac{\partial\eta}{\partial\varphi} \tag{1.3}$$

The system of equations (1.3) is to be integrated. The function h will be determined during this integration. The constants in the h function are to be selected in such a way, that in a certain interval η the function h will approximate to the function b in the equalities (1.2). If we succeed, the solution of the system (1.3) will be an approximate solution of Chaplygin's equations in that range of values η, wherein the function h approximates to the function b.

To integrate the auxiliary system of equations, we eliminate the angle θ from the system (1.3):

$$\frac{\partial}{\partial\varphi}\left(h\eta\frac{\partial\eta}{\partial\varphi}\right) + \frac{\partial}{\partial\psi}\left(\frac{1}{h}\frac{\partial\eta}{\partial\psi}\right) = 0 \tag{1.4}$$

We seek an integral of this equation in the form

$$\Phi(\eta) = e^{c\varphi}F(\psi) \tag{1.5}$$

where $\phi(\eta)$ and $F(\psi)$ are functions to be determined, c is a constant.

Putting into equations (1.4) the values of the derivatives obtained with the help of (1.5), we obtain, after carrying out the differentiation

$$\left[\left(\frac{h\eta}{\Phi'}\right)' + \frac{h\eta}{\Phi}\right]c^2 + \left(\frac{1}{h\Phi'}\right)'\frac{1}{F^2}\left(\frac{dF}{d\psi}\right)^2 + \frac{1}{h\Phi}\frac{1}{F}\frac{d^2F}{d\psi^2} = 0 \tag{1.6}$$

The prime here and further on means defferentiation with respect to η. Assuming

$$\left(\frac{1}{h\Phi'}\right)' = 0, \qquad \left(\frac{h\eta}{\Phi'}\right)' + \frac{h\eta}{\Phi} = \frac{c_1^2}{h\Phi} \tag{1.7}$$

where c_1 is a constant, this results in

$$\frac{d^2F}{d\psi^2} + c_1^2c^2F = 0 \tag{1.8}$$

The three unknown functions Φ, h, F, can be determined from equations (1.7) and (1.8).

Particular solutions of equation (1.8) are the functions:

$$F = c_2\cos c_1c\psi \tag{1.9}$$

The first of equations (1.7) gives

$$h\Phi' = c_3 = \text{const} \tag{1.10}$$

Introducing this, the second of equations (1.7) gives

$$(h^2\eta)' = \frac{c}{\Phi h}(c_1^2 - h^2\eta), \qquad \text{or} \qquad \Phi = \frac{c_4}{h^2\eta - c_1^2} \qquad (c_4 = \text{const}) \tag{1.11}$$

We introduce a new function

$$z = h\sqrt{-\eta} \tag{1.12}$$

(In what follows we shall be interested in negative values of η).

Putting the expression Φ', obtained from (1.11), into (1.10) we obtain

$$\frac{c_3}{c_4}\sqrt{-\eta}\, d(-\eta) = -\frac{2z^2dz}{(z^2+c_1^2)^2} \tag{1.13}$$

After carrying out the integration and substituting z, in agreement with (1.12), an expression is obtained, which determines the function h (for $c_3 = 1$):

$$\frac{h\sqrt{-\eta}}{c_1^2 - h^2\eta} + \frac{1}{c_1}\operatorname{arc\,tg}\frac{c_1}{h\sqrt{-\eta}} = \frac{2}{3c_4}(-\eta)^{3/2} + c_5 \tag{1.14}$$

Thus, the integral (1.5) of the equation (1.4) has the form

$$\Phi = c_2e^{c\varphi}\cos c_1c\psi \tag{1.15}$$

wherein Φ is determined by equation (1.11). At the same time function h of equation (1.4) is determined by relation (1.14).

In order to complete the integration of the auxiliary system (1.3), the expression for the angle of inclination of the velocity vector has to be obtained.

Returning to the system (1.3), we get

$$\theta = -\int \frac{1}{h}\frac{\partial\eta}{\partial\psi}\,d\varphi + U(\psi)$$

where $U(\psi)$ is a function to be determined.

The expression for the angle θ is obtained by taking $\partial\eta/\partial\psi$ from (1.15 and comparing with (1.10):

$$\theta = c_1c_2e^{c\varphi}\sin cc_1\psi + U(\psi)$$

After substitution of $\partial\eta/\partial\phi$ and $\partial\theta/\partial\psi$ into the second equation of the system (1.3) the function $U(\psi)$ is obtained by integration, and hence the required function θ:

$$\theta = c_1c_2e^{c\varphi}\sin cc_1\psi + cc_4\psi \tag{1.16}$$

2. Approximate integration of Chaplygin's equations. Expressions (1.15) and (1.16) are integrals of the system (1.3) in the case when the function h is determined by equation (1.14). However, we are interested in the system (1.1) which, instead of the function h contains the function b, defined by (1.2).

Therefore, the constants c_1, c_4 and c_5 in the expression (1.14), defining the function h, are to be selected in such a way, that the function h will approximate to the function b. Then the integrals of the auxiliary system of equations (1.3) may be considered to be the approximate solution of Chaplygin's equations.

We are interested in the solution of Chaplygin's equations for the supersonic zone of the nozzle, adjacent to the transition line. Therefore we will approximate the function b in the region of negative values of η, beginning with $\eta = 0$ ($\bar{v} = 1.0$).

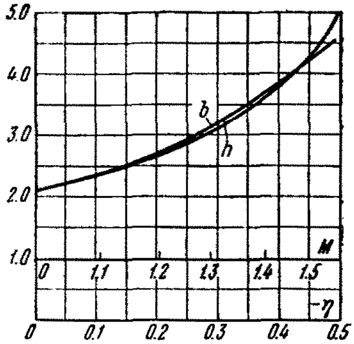

Fig. 1.

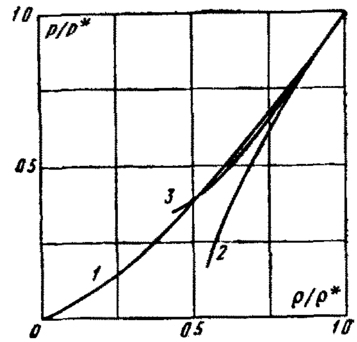

Fig. 2.

To select the constants, we require that equations (1.11), (1.14), (1.15) be satisfied at the center of the nozzle, where $\phi = \psi = 0$ and $\eta = \theta = 0$.

Indicating the values of functions on the transition line, by an asterisk (*), (remembering that $c_3 = 1$), we obtain

$$c_1^2 = \Phi_* h_*^3, \qquad c_2 = \Phi_*, \qquad c_4 = -\Phi_*^2 h_*^3, \qquad c_5 = \frac{\pi}{2\sqrt{\Phi_* h_*^3}} \tag{2.1}$$

Hence, the integrals of the auxiliary system of equations for the case discussed are:

$$t = e^{c\varphi} \cos c\sqrt{\Phi_* h_*^3}\,\psi \tag{2.2}$$

$$\theta = \sqrt{\Phi_*^3 h_*^3}\,(e^{c\varphi} \sin c\sqrt{\Phi_* h_*^3}\psi - c\sqrt{\Phi_* h_*^3}\psi)$$

Here

$$t = \frac{\Phi}{\Phi_*} = \frac{\Phi_* h_*^3}{\Phi_* h_*^3 - \eta h^2} \tag{2.3}$$

$$\frac{h\sqrt{-\eta}}{\Phi_* h_*^3 - h^2\eta} + \frac{1}{\sqrt{\Phi_* h_*^3}} \operatorname{arctg} \frac{1}{h}\sqrt{\frac{\Phi_* h_*^3}{-\eta}} = \frac{\pi}{2\sqrt{\Phi_* h_*^3}} - \frac{2}{3}\,\frac{1}{\Phi_*^2 h_*^3}(-\eta)^{3/2} \tag{2.4}$$

Computations show that, when $h_* = b_* = 2.11$ and $\Phi_* = 0.2$, the function h approximates quite well (Fig.1) to the function b between the limits $0 > \eta > -0.45$, i.e. for

$$1 \leqslant M < 1.5 \tag{2.5}$$

As is known, replacement of the function b by the function h is equivalent to the replacement of the adiabat by another relationship between the gas density and pressure. This relationship can be easily established when the function h is known. Simple transformations give, for the case discussed

$$\frac{P}{P_*} = 1 - 1.4\left(1 - \frac{\rho}{\rho_*}\right) + 0.295\left(1 - \frac{\rho}{\rho_*}\right)^2 + 0.012\left(1 - \frac{\rho}{\rho_*}\right)^3 + \cdots$$

The analogous series for the adiabatic relationship is:

$$\frac{P}{P_*} = 1 - 1.4\left(1 - \frac{\rho}{\rho_*}\right) + 0.280\left(1 - \frac{\rho}{\rho_*}\right)^2 + 0.056\left(1 - \frac{\rho}{\rho_*}\right)^3 + \cdots$$

Comparison of both expressions and the results of computations (Fig.2) show that over the range (2.5), approximation of the adiabat by the relationship between density and pressure, corresponding to the function h of (2.4), can be considered satisfactory. In order to prove that the functions (2.2) really represent the flow in the initial portion of the supersonic part of the Laval nozzle, which has a subsonic part of the type given in (0.1), it is necessary to check:

(1) that equations (2.2) satisfy the symmetry conditions with respect to the streamline $\psi = 0$:

$$\eta(\varphi, \psi) = \eta(\varphi, -\psi), \qquad \theta(\varphi, \psi) = -\theta(\varphi, -\psi), \qquad \theta(\varphi, 0) = 0$$

(2) that the flow given by equations (2.2) joins on the transition line with the flow given by the function (0.1). The first is obvious. The following section is concerned with the proof of the second condition.

3. Expansion in series of a stream function and its derivatives. In Ref. [1] Frankl' demonstrated that the stream function (0.1) representing the subsonic part of the nozzle, and also its derivatives, can be given on the transition line by the series:

$$\begin{aligned}
\psi_* &= -B_2 \frac{\pi (2B)^{1/3}}{\Gamma(4/3)\, 3^{1/2}} \theta^{1/3} + D_1\theta + D_2\theta^{5/3} + \cdots + D_m\theta^{\frac{2m+1}{3}} + \cdots \\
\left(\frac{\partial\psi}{\partial\eta}\right)_* &= -B_2 \frac{\pi (2B)^{1/3}}{\Gamma(4/3)\, 3^{7/6}} \theta^{-1/3} + E_1\theta^{1/3} + E_2\theta + \cdots + E_m\theta^{\frac{2m-1}{3}} + \cdots \\
\left(\frac{\partial\psi}{\partial\theta}\right)_* &= -B_2 \frac{\pi (2B)^{4/3}}{\Gamma(4/3)\, 3^{3/2}} \theta^{-2/3} + D_1 + \frac{5}{3} D_2\theta^{2/3} + \cdots + \frac{2m+1}{3} D_m\theta^{\frac{2m-2}{3}} + \cdots \\
\left(\frac{\partial^2\psi}{\partial\theta\,\partial\eta}\right)_* &= B_2 \frac{\pi (2B)^{-1/3}}{\Gamma(4/3)\, 3^{13/6}} \theta^{-4/3} + \frac{1}{3} E_1\theta^{-2/3} + E_2 + \cdots + \frac{2m-1}{3} E_m\theta^{\frac{2m-4}{3}} + \cdots
\end{aligned} \tag{3.1}$$

where D and E are constants.

We will show that the stream function defined by equations (2.2) and representing the initial portion of the supersonic part of the nozzle, and also its derivatives, can be given by a series of similar form on the transition line.

Equations (2.2) are used, eliminating ϕ. Then

$$\theta = \sqrt{\Phi_*^3 h_*^3}\,(t \operatorname{tg} c\sqrt{\Phi_* h_*^3}\psi - c\sqrt{\Phi_* h_*^3}\psi) \tag{3.2}$$

Taking into consideration that $t = 1.0$ on the transition line, and expanding $\operatorname{tg} c\sqrt{(\Phi_* h_*^3 \psi)}$ in series, θ can be represented by the following series:

$$\theta = r\,(a_3\delta^3 + a_5\delta^5 + a_7\delta^3 + \cdots) \qquad (r = \sqrt{\Phi_*^3 h_*^3}, \qquad \delta = c\sqrt{\Phi_* h_*^3}\psi_*) \tag{3.3}$$

The function ψ_* can be expanded in a series of powers of $\theta^{1/3}$. Raising both parts of (3.3) by the power 1/3, we obtain

$$\theta^{1/3} = r^{1/3}\delta\,(a_3 + a_5\delta^2 + a_7\delta^4 + \cdots)^{1/3} = g_1\delta + g_2\delta^3 + g_3\delta^5 + \cdots$$

If the last series is inverted, and its first coefficient computed, then

$$\psi_* = \frac{3^{1/3}}{c\Phi_* h_*^2}\,\theta^{1/3} + N_1\theta + N_2\theta^{5/3} + \cdots + N_m\theta^{\frac{2m+1}{3}} + \ldots \tag{3.4}$$

Hence

$$\left(\frac{\partial\psi}{\partial\theta}\right)_* = \frac{1}{3^{2/3} c\Phi_* h_*^2}\,\theta^{-2/3} + N_1 + \frac{5}{3}N_2\theta^{2/3} + \ldots + \frac{2m+1}{3}N_m\theta^{\frac{2m-2}{3}} + \ldots \tag{3.5}$$

To determine a series for $\partial\psi/\partial\eta$, we use equation (3.2). Differentiating first by η, and then by θ, and assuming $t = 0$, we find that on the transition line

$$\left(\frac{\partial\psi}{\partial\eta}\right)_* = \frac{1}{c\Phi_* h_*^3}\sqrt{\frac{h_*}{\Phi_*}}\operatorname{ctg} c\sqrt{\Phi_* h_*^3}\,\psi_*, \qquad \left(\frac{\partial\psi}{\partial\theta}\right)_* = \frac{1}{c\Phi_*^2 h_*^3}\operatorname{ctg}^2 c\sqrt{\Phi_* h_*^3}\,\psi_*$$

From these equations it follows that

$$\left(\frac{\partial\psi}{\partial\eta}\right)_* = \sqrt{\frac{1}{c\Phi_* h_*^2}\left(\frac{\partial\psi}{\partial\theta}\right)_*}$$

Using (3.5) it is simple to find

$$\left(\frac{\partial\psi}{\partial\eta}\right)_* = \frac{1}{3^{1/3} c\Phi_* h_*^2}\,\theta^{-1/3} + J_1\theta^{1/3} + J_2\theta + \ldots + J_m\theta^{\frac{2m-1}{3}} + \ldots$$

$$\left(\frac{\partial^2\psi}{\partial\theta\partial\eta}\right)_* = -\frac{1}{3^{4/3} c\Phi_* h_*^2}\,\theta^{-4/3} + \frac{1}{3}J_1\theta^{-2/3} + J_2 + \ldots + \frac{2m-1}{3}J_m\theta^{\frac{2m-4}{3}} + \ldots$$

Therefore, the stream function represented by equations (2.2) and its derivatives really can be expressed by a series of the same form as the stream function (0.1).

The solution obtained by Fal'kovich [2] for the transition zone of the nozzle can be derived from the solution (2.2) as its first approximation. To show this, we expand the functions in (2.2) in series and take only the first two members of each series. Applying (2.3) and replacing $h_{\text{ц}}$ by its equivalent b^*, we have

$$1+\frac{1}{\Phi_* b_*^3}\eta=(1+c\varphi)\left(1-\frac{1}{2}c^2\Phi_* b_*^3\psi^2\right)$$

$$\theta=\sqrt{\Phi_*^3 b_*^3}\left[(1+c\varphi)(c\sqrt{\Phi_* b_*^3}\,\psi-\frac{1}{6}c^3\Phi_*^{3/2}b_*^{9/2}\psi^3)-c\sqrt{\Phi_* b_*^3}\,\psi\right]$$

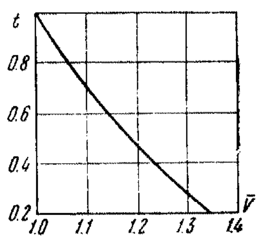

Fig. 3.

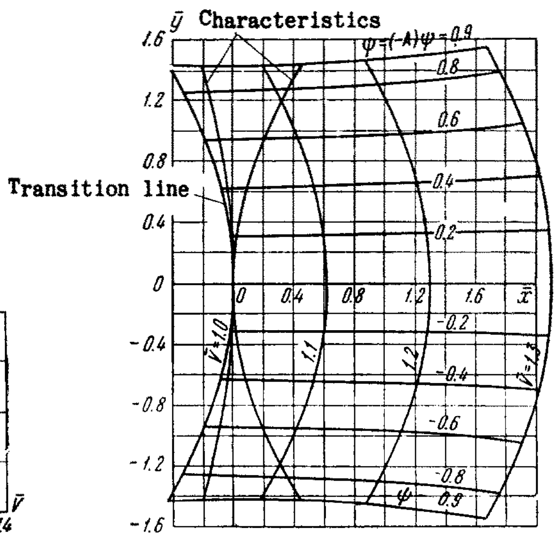

Fig. 4.

After opening parentheses and neglecting the term containing $\phi\psi^2$ in the first equation and the term with $\phi\psi^3$ in the second equation, the transformation gives:

$$\eta=c\Phi_* b_*\left(\varphi-\frac{1}{2}c\Phi_* b_*^3\psi^2\right),\qquad \theta=c^2\Phi_*^2 b_*^3\left(\varphi\psi-\frac{1}{6}c\Phi_* b_*^3\psi^3\right)$$

These expressions coincide with the solution in Ref. [2]. The constant A, introduced in Ref. [2], is linked with the constant c by the relation $A = c\Phi_* b_*^3$. This means that the constant c has the same meaning as the constant A in Ref. [2].

4. Construction of the flow in the initial portion of the supersonic part of the nozzle. Equations (2.2) make possible the construction of a nozzle flow in the $\phi\psi$ plane. Transformation to the xy plane can be accomplished in the usual way with the help of the relations

$$dx = \frac{1}{v}\cos\theta\, d\varphi - \frac{\rho_0}{\rho v}\sin\theta d\varphi, \qquad dy = \frac{1}{v}\sin\theta d\varphi + \frac{\rho_0}{\rho v}\cos\theta d\psi$$

The following formulas are obtained for the coordinates of streamlines in the transition zone of the nozzle:

$$\bar{x} - \bar{x}_* = \Phi_* b_*^3 \int_t^1 \frac{1}{vt}\cos\theta dt \qquad \left(\bar{x} = \frac{x}{-Aa_*}\right)$$

$$\bar{y} - \bar{y}_* = \Phi_* b_*^3 \int_t^1 \frac{1}{vt}\sin\theta dt \qquad \left(\bar{y} = \frac{y}{-Aa_*}\right)$$

In these formulas x_*, y_* are the coordinates of the intersection point of a streamline with the transition line, t is the function defined by equation (2.3) and shown in Fig. 3.

Similar expressions can be derived for other characteristic curves: lines of equal velocity characteristics, and others.

When dimensionless coordinates are used the flow in different nozzles (different values of A) can be plotted on the same diagram.

Results of the computation of the streamlines and lines of constant velocity are presented in Fig. 4.

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